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REGULAR REPRESENTATIONS OF DIRICHLET SPACES

BY
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Abstract. We construct a regular and a strongly regular Dirichlet space which are equivalent to a given Dirichlet space in the sense that their associated function algebras are isomorphic and isometric. There is an appropriate strong Markov process called a Ray process on the underlying space of each strongly regular Dirichlet space.

1. **Introduction.** A. Beurling and J. Deny [1] introduced the notion of Dirichlet spaces and developed the general theory of kernel-free potentials. Recently the author [6] adopted Dirichlet spaces relative to L^2-spaces (we will call them L^2-Dirichlet spaces or D-spaces as an abbreviation) to describe boundary conditions for multidimensional Brownian motions.

A D-space is a certain space of functions that are defined on an underlying measure space (X, m). When (X, m) is fixed, there is a one-to-one correspondence between the set of all symmetric sub-Markov resolvent operators on $L^2(X; m)$ and the set of all D-spaces. In particular, any sub-Markov resolvent kernel on X which is symmetric with respect to m generates a D-space. The present paper and the subsequent one [9] concern the problem of whether conversely any D-space guarantees the existence of a suitable strong Markov process or not.

The present paper aims at constructing a regular and a strongly regular D-space which are equivalent to a given D-space. A D-space is called regular if it densely contains sufficiently many continuous functions vanishing at infinity on its underlying space. There corresponds a potential theory of a type of Beurling-Deny to each regular D-space. A strongly regular D-space is a regular one which is generated by a Ray resolvent kernel. According to D. Ray [15], there is a right continuous strong Markov process on the underlying space of each strongly regular D-space.

Suppose that we are given a D-space with underlying space (X, m). Theorem 2 in §5 states that there exists then a regular D-space with some modified underlying space (X', m') in such a way that these two D-spaces are equivalent to each other as function spaces. The latter D-space will be called a *regular representation* of the given one. The regular representation will be carried out depending on a subalgebra L of $L^\infty(X; m)$ satisfying a certain condition denoted by (C). Actually we will take as X' the space of all regular maximal ideals of L.

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There are generally many possibilities to find L satisfying (C). In §6, a special L possessing an additional property denoted by (R) will be constructed by making use of the method of F. Knight [11] and H. Kunita and T. Watanabe [12]. We can regard the condition (R) as a generalization of Ray's hypothesis for a sub-Markov resolvent [15]. Theorem 3 in §6 asserts that the regular representation with respect to such an L turns out to be a strongly regular D-space.

§3 consists of typical examples of D-spaces related to the multidimensional Brownian motion. Those D-spaces except for the last example took the fundamental roles in the investigations of boundary problems by J. L. Doob [4] and by the author [5], [6]. The last example is a rather sophisticated one of regular D-spaces(1). Much stress on the roles of regular ones will be laid in [9].

The appendix is referred to only in §3.

2. **Basic properties of D-spaces.**

DEFINITION 2.1. We call $(X, m, \mathscr{F}, \mathscr{E})$ an *L^2-Dirichlet space* (or a *D-space*, for short) if the following conditions are satisfied.

(D.1) X is a locally compact, Hausdorff, and separable space. m is a Radon measure on X.

(D.2) $\mathscr{F}$ is a linear subspace of the real $L^2(X)=L^2(X; m)$, two functions of $\mathscr{F}$ being identified if they coincide m-a.e. on X. $\mathscr{E}$ is a symmetric nonnegative definite bilinear form on $\mathscr{F}$ and, for each $\alpha>0$, $\mathscr{F}$ is a real Hilbert space with respect to the inner product

$$\mathscr{E}^{\alpha}(u, v) = \mathscr{E}(u, v)+\alpha(u, v)_X, \qquad u, v \in \mathscr{F}, \tag{2.1}$$

where $(u, v)_X$ denotes the inner product of $L^2(X)$.

(D.3) Every normal contraction operates on $(\mathscr{F}, \mathscr{E})$: if $u \in \mathscr{F}$ and a m-measurable function v satisfies inequalities

$$|v(x)| \leqq |u(x)|, \qquad |v(x)-v(y)| \leqq |u(x)-u(y)|$$

m-a.e. on X, then $v \in \mathscr{F}$ and $\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$.

The present definition of D-space was given in [6]. (X, m) is called the *underlying space* of the D-space. According to §2 of [6], let us state a theorem about a one-to-one correspondence between D-spaces and L^2-resolvents.

DEFINITION 2.2. Let (X, m) satisfy condition (D.1). A system $\{G_\alpha, \alpha>0\}$ of linear, bounded and symmetric operators on $L^2(X)$ is called an *L^2-resolvent* if it has the following properties.

(G.1) Sub-Markov property: if $u \in L^2(X)$ and $0 \leqq u \leqq 1$ m-a.e. then $0 \leqq \alpha G_\alpha u \leqq 1$ m-a.e., for any $\alpha>0$.

(G.2) Resolvent equation: $G_\alpha - G_\beta+(\alpha-\beta)G_\alpha G_\beta=0$, $\alpha, \beta>0$.

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(1) N. Ikeda suggested to the author the last example of §3 and theorem of the appendix.

THEOREM 1. *Let us fix* (X, m) *satisfying condition* (D.1). *For a given D-space* $(\mathscr{F}, \mathscr{E})$ *with underlying space* (X, m), *there exists a unique* L^2*-resolvent* $\{G_\alpha, \alpha>0\}$ *on* $L^2(X)$ *satisfying the equation*

$$\mathscr{E}^\alpha(G_\alpha u, v) = (u, v)_X \tag{2.2}$$

for any $v \in \mathscr{F}$, *where* $\alpha>0$ *and* $u \in L^2(X)$ *are arbitrarily fixed. Conversely, for a given* L^2*-resolvent* $\{G_\alpha, \alpha>0\}$ *on* $L^2(X)$, *a D-space is defined by*

$$\mathscr{F} = \left\{u \in L^2(X);\ \lim_{\beta\to+\infty} \beta(u-\beta G_\beta u, u)_X < +\infty\right\}, \tag{2.3}$$

$$\mathscr{E}(u, v) = \lim_{\beta\to+\infty} \beta(u-\beta G_\beta u, v)_X, \qquad u, v \in \mathscr{F}. \tag{2.4}$$

The correspondence defined by (2.2) *and that defined by* (2.3) *and* (2.4) *are reciprocal to each other.*

REMARK 2.1. (i) The proof of Theorem 1 was sketched in §2 of [6]. The essential ideas for the proof can be found in Beurling-Deny [1] and Deny [2]. So far as this theorem and the next lemma are concerned, condition (D.1) for (X, m) can be much weakened. These have been proved in [7] without the separability assumption for X (see also [8]). T. Shiga and T. Watanabe [16] gave a detailed proof of Theorem 1 under the assumption that, instead of (D.1), the underlying space (X, m) is merely a σ-finite measure space.

(ii) Condition (D.3) in the definition of D-space can be replaced with the following apparently weaker but equivalent condition (D.3)′ [16].

(D.3)′ Every unit contraction operates on $(\mathscr{F}, \mathscr{E})$: if $u \in \mathscr{F}$ then $v=(0\vee u)\wedge 1$ is also in $\mathscr{F}$ and $\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$. Here, the lattice operations $\vee$ and $\wedge$ for functions on X are defined by $(u_1 \vee u_2)(x)=\max(u_1(x), u_2(x))$ and $u_1\wedge u_2=-((-u_1)\vee(-u_2))$.

The next lemma states the basic properties of D-spaces which we need in the later discussions. Notice that, for a D-space, $\mathscr{E}^\alpha$ and $\mathscr{E}^\beta$ define equivalent metrics on $\mathscr{F}$ for any $\alpha, \beta>0$.

LEMMA 2.1. *Let* $(X, m, \mathscr{F}, \mathscr{E})$ *be a D-space and* $\{G_\alpha, \alpha>0\}$ *be its associated* L^2*-resolvent. Fix an* $\alpha_0>0$.

(i) *If S is a dense subset of* $L^2(X)$, *then, for any* $\alpha>0$, $G_\alpha(S)$ *is dense in* $\mathscr{F}$ *with respect to metric* $\mathscr{E}^{\alpha_0}$.

(ii) *For* $u, v \in \mathscr{F}$,

$$\mathscr{E}^\alpha(u, v) = \lim_{\beta\to+\infty} \beta(u-\beta G_{\beta+\alpha}u, v)_X. \tag{2.5}$$

(iii) *For any* $u \in \mathscr{F}$, $\lim_{\beta\to+\infty}\beta G_\beta u=u$ *strongly in norm* $\mathscr{E}^{\alpha_0}$ *and hence strongly in* $L^2(X)$ *sense.*

(iv) $\mathscr{F}$ *is a function lattice: if* $u, v \in \mathscr{F}$, *then* $u\vee v$, $u\wedge v \in \mathscr{F}$. *Further* $u\wedge 1 \in \mathscr{F}$ *for* $u \in \mathscr{F}$.

(v) *If u and v are both in $\mathscr{F}$ and m-essentially bounded, then the product $u \cdot v$ is also in $\mathscr{F}$.*

(vi) *For $u \in \mathscr{F}$, put $u_n = ((-n) \vee u) \wedge n$.*
Then $\lim_{n\to+\infty} u_n = u$ strongly in norm $\mathscr{E}^{\alpha_0}$.

Proof. (i) is a consequence of the equation (2.2).

(ii) is a consequence of Lemma 1 of [8].

(iii) For $\beta > \alpha_0$,

$$\begin{aligned}\mathscr{E}^{\alpha_0}(\beta G_\beta u - u, \beta G_\beta u - u) &\leqq \mathscr{E}^{\beta}(\beta G_\beta u - u, \beta G_\beta u - u)\\ &= \beta^2(\beta G_\beta u, u)_X - 2\beta(u, u)_X + \mathscr{E}^\beta(u, u)\\ &= -\beta(u - \beta G_\beta u, u)_X + \mathscr{E}(u, u) \to 0, \qquad \beta \to +\infty.\end{aligned}$$

(iv) Since $|u|$ and $u \wedge 1$ are normal contractions of u, they are in $\mathscr{F}$ if u is. Note that

$$u \vee v = \tfrac{1}{2}((u+v) + |u-v|), \qquad u \wedge v = \tfrac{1}{2}((u+v) - |u-v|).$$

(v) If $u \in \mathscr{F}$ and $|u| \leqq M$ m-a.e. for some constant M, then u^2 is a normal contraction of $2Mu$ and hence $u^2 \in \mathscr{F}$. Note that $u \cdot v = \frac{1}{4}((u+v)^2 - (u-v)^2)$.

(vi) By Lemma 2.1 of [6], $\mathscr{E}^{\alpha_0}(u_n, u_n)$ increases to $\mathscr{E}^{\alpha_0}(u, u)$ as n tends to infinity. On the other hand,

$$\mathscr{E}^{\alpha_0}(u_n, G_{\alpha_0} w) = (u_n, w)_X \xrightarrow[n\to+\infty]{} (u, w)_X = \mathscr{E}^{\alpha_0}(u, G_{\alpha_0} w)$$

for any $w \in L^2(X)$. These facts combined with the first statement of this lemma imply that u_n converges to u weakly and after all strongly with respect to the inner product $\mathscr{E}^{\alpha_0}$.

We will now give definitions and remarks concerning regularity of D-spaces. For a locally compact space X, denote by $C(X)$ (resp. $C_0(X)$) the space of all continuous functions vanishing at infinity (resp. with compact supports). $C^+(X)$ (resp. $C_0^+(X)$) will denote the set of all nonnegative elements of $C(X)$ (resp. $C_0(X)$). We say a measure m on X to be *everywhere dense* if $m(E)$ is not zero for any nonempty open set $E \subset X$.

DEFINITION 2.3. A D-space $(X, m, \mathscr{F}, \mathscr{E})$ is called *regular* if m is everywhere dense and $\mathscr{F} \cap C(X)$ is dense both in $\mathscr{F}$ with norm $\mathscr{E}^{\alpha_0}$ and in $C(X)$ with uniform norm. Here, $\alpha_0 > 0$ is arbitrarily fixed.

Next, consider (X, m) satisfying condition (D.1). For a sub-Markov resolvent kernel(2) $\{G_\alpha(x, E), \alpha > 0\}$ on X, we set

$$G_\alpha u(x) = \int_X G_\alpha(x, dy) u(y), \qquad u \in C(X). \tag{2.6}$$

(2) $G_\alpha(x, E)$ is called a kernel on X if, for a fixed $x \in E$, $G_\alpha(x, \cdot)$ is a Borel measure on X and, for a fixed Borel set $E \subset X$, $G_\alpha(\cdot, E)$ is a measurable function on X.

DEFINITION 2.4. (i) A sub-Markov resolvent kernel $\{G_\alpha(x, E), \alpha>0\}$ on X is called *m-symmetric* if

$$\int_X G_\alpha u(x)\cdot v(x)m(dx) = \int_X u(x)\cdot G_\alpha v(x)m(dx) \leqq +\infty$$

for any $u, v \in C^+(X)$. (ii) A sub-Markov resolvent kernel $\{G_\alpha(x, E), \alpha>0\}$ on X is called a *Ray resolvent* if it satisfies the following conditions.

(R.a) $G_\alpha(C(X))\subset C(X)$ for any $\alpha>0$.

(R.b) There exists a countable subcollection C_1 of $C^+(X)$ such that (α) C_1 separates points of X, and, for any $x \in X$, there exists a $u \in C_1$ whose value at x is not zero, (β) for some $\alpha_0>0$, every function $u \in C_1$ satisfies the inequality $\beta G_{\alpha_0+\beta}u \leqq u$, $\beta>0$.

Consider any m-symmetric sub-Markov resolvent kernel $\{G_\alpha(x, E), \alpha>0\}$ on X. It satisfies the inequality $(\alpha G_\alpha u, \alpha G_\alpha u)_X \leqq (u, u)_X$ for all $u \in L^2(X; m) \cap C(X)$ [16]. Therefore it determines a unique L^2-resolvent. The Dirichlet space associated with this L^2-resolvent will be said *to be generated by the resolvent kernel* $\{G_\alpha(x, E), \alpha>0\}$.

We will say the set C_1 appearing in the definition of Ray resolvent *to be attached to the given Ray resolvent.*

DEFINITION 2.5. A D-space $(X, m, \mathcal{F}, \mathcal{E})$ is called *strongly regular* if m is everywhere dense on X, $(\mathcal{F}, \mathcal{E})$ is generated by an m-symmetric Ray resolvent on X and $\mathcal{F} \cap C(X)$ contains the set C_1 attached to this Ray resolvent.

REMARK 2.2. (i) A strongly regular D-space is regular. To see this, let $(X, m, \mathcal{F}, \mathcal{E})$ be a strongly regular D-space and $\{G_\alpha(x, E), \alpha>0\}$ be its associated Ray resolvent. $\mathcal{F} \cap C(X)$ contains $G_\alpha(L^2(X) \cap C(X))$, which is dense in $(\mathcal{F}, \mathcal{E}^{\alpha_0})$ by virtue of Lemma 2.1(i). Owing to the fifth statement of the lemma, $\mathcal{F} \cap C(X)$ is a function algebra. Since it contains the set C_1 attached to $\{G_\alpha, \alpha>0\}$, it is dense in $C(X)$ by Stone-Weierstrass theorem.

(ii) Consider a Ray resolvent $\{G_\alpha(x, E), \alpha>0\}$ on a locally compact Hausdorff separable space X. Let $\bar{X}=X \cup \{\infty\}$ be the one point compactification of X if X is not compact. If X is compact, let $\{\infty\}$ be an isolated point. Define a new kernel $\{\bar{G}_\alpha(x, E), \alpha>0\}$ on $\bar{X}$ by $\bar{G}_\alpha(x, E)=G_\alpha(x, E \cap X)+((1-\alpha G_\alpha(x, X))/\alpha)\delta_{\{\infty\}}(E)$, $x \in X$, $\bar{G}_\alpha(\{\infty\}, E)=(1/\alpha)\delta_{\{\infty\}}(E)$. Then $\{\bar{G}_\alpha, \alpha>0\}$ is a conservative Ray resolvent on the compactum $\bar{X}$. By Ray's theory [15], [12], this defines on $\bar{X}$ a right continuous conservative strong Markov process for which the point $\{\infty\}$ is a trap. Thus, we obtain a right continuous strong Markov process $(X_t, \zeta, P_x, x \in X)$ on X such that

$$G_\alpha(x, E) = E_x\left(\int_0^\zeta e^{-\alpha t}\chi_E(X_t)\, dt\right), \tag{2.7}$$

χ_E being the indicator function of the Borel set E. We will call the process on X so obtained the *Ray process* associated with the Ray resolvent $\{G_\alpha, \alpha>0\}$ on X.

There is a Ray process on the underlying space of any strongly regular D-space.

3. **Examples.** Denote by D a domain of Euclidean N-space R^N ($N \geqq 1$).

EXAMPLE 1. Let us put

$$\mathscr{E}^1_{L^2}(D) = \{u;\, u \in L^2(D),\ \partial u/\partial x_i \in L^2(D),\ i = 1, 2, \ldots, N\},$$

$$(u, v)_{D,1} = \frac{1}{2}\int_D{}' \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\, dx.$$

Here, derivatives are taken in Schwartz distribution sense and dx denotes the Lebesgue measure on R^N.

$(D, dx, \mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1})$ is a D-space in our sense. Condition (D.3)′ for this space can be verified easily (see Proposition A.1 of [16] or Théorème 3.1 of [3]). This space is not regular except when it coincides with $\mathscr{D}^1_{L^2}(D)$ of the next example. Denote by $\bar{D}$ the closure of D in R^N. Let $C^\infty(\bar{D})$ be the space of restrictions to $\bar{D}$ of functions which are infinitely differentiable on R^N. If $\partial D = \bar{D} - D$ is a closed hypersurface of class C^1, then $\mathscr{E}^1_{L^2}(D) \cap C^\infty(\bar{D})$ is dense in $\mathscr{E}^1_{L^2}(D)$ [14]. Therefore, in this case, $(\bar{D}, dx, \mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1})$(3) is a regular D-space. When D is bounded, the space $(\mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1})$ is generated by the continuous resolvent density constructed in [5] and in §8(I) of [6].

EXAMPLE 2. Denote by $C_0^\infty(D)$ the space of infinitely differentiable functions on D with compact supports. Let $\mathscr{D}^1_{L^2}(D)$ be the closure of $C_0^\infty(D)$ in

$$(\mathscr{E}^1_{L^2}(D), (\ ,\ )_{D,1} + (\ ,\ )_D).$$

$(D, dx, \mathscr{D}^1_{L^2}(D), (\ ,\ )_{D,1})$ is a regular D-space. Since $\mathscr{D}^1_{L^2}(D)$ coincides with the completion of $\mathscr{E}^1_{L^2}(D) \cap C_0(D)$ with respect to metric $(\ ,\ )_{D,1} + (\ ,\ )_D$, we can apply Corollary 3 of Appendix to show that it is a regular D-space. It is generated by a continuous resolvent density of the absorbing barrier Brownian motion on D [6]. It is strongly regular when each point of the boundary ∂D is regular with respect to the Dirichlet problem for D.

EXAMPLE 3. Let M be the Martin boundary of the domain D and μ be the harmonic measure on M with respect to a reference point x_0 of D. J. L. Doob [4] introduced the space $\boldsymbol{H}_h'$ of measurable functions φ on M for which the integral

$$\boldsymbol{D}_M(\varphi, \varphi) = \frac{q}{4}\int_M \int_M (\varphi(\xi) - \varphi(\eta))^2 \theta(\xi, \eta)\mu(d\xi)\mu(d\eta)$$

is finite. Here, $\theta(\xi, \eta)$ is Naim's kernel on M and q is 2π if $N=2$ or the product of $N-2$ and the unit ball boundary area if $N>2$. It was proved in [4] that $\boldsymbol{H}_h' \subset L^2(M; \mu)$. We can easily see that $(M, \mu, \boldsymbol{H}_h', \boldsymbol{D}_M(\ ,\ ))$ is a D-space. This is regular when D is a disk (§18 of [4]). Let $\boldsymbol{H}_h$ be the space of all harmonic functions on D with finite integrals $(u, u)_{D,1}$. Then, $(\boldsymbol{H}_h', \boldsymbol{D}_M(\ ,\ ))$ is the trace on M of the space $(\boldsymbol{H}_h, (\ ,\ )_{D,1})$ in the following sense: each function u of $\boldsymbol{H}_h$ has a fine boundary

(3) We regard here $\mathscr{E}^1_{L^2}(D)$ as a subspace of $L^2(\bar{D})$ $(=L^2(D))$. See Remark 5.2.

limit function γu in $\boldsymbol{H}'_h$ and γ define a unitary map from $(\boldsymbol{H}_h, (\ , \ )_{D,1})$ onto $(\boldsymbol{H}'_h, \boldsymbol{D}_M)$. This is the reason why functions of H'_h were called in [4] BLD boundary functions.

A modification of the space $(\boldsymbol{H}'_h, \boldsymbol{D}_M)$ was introduced in [6] in order to describe the space of all α-harmonic functions of $\mathscr{E}^1_{L^2}(D)$. Suppose that D is bounded. Let $U_\alpha(\xi, \eta)$ and $U(\xi, \eta)$ be Feller kernels on M. $U(\xi, \eta)$ is equal to $(q/2)\cdot\theta(\xi, \eta)$ μ-a.e. Denote by μ' the measure $U_1 1\cdot\mu$ on M and put $\boldsymbol{H}_M=\boldsymbol{H}'_h \cap L^2(M; \mu')$. Then, $(M\ \mu'\ \boldsymbol{H}_M, \boldsymbol{D}_M)$ is a D-space. By virtue of Lemma 3.1 and equality (3.21) of [6], it is clear that $(M, \mu', \boldsymbol{H}_M, \boldsymbol{D}_M^{[\alpha]})$ is also a D-space for each $\alpha>0$, where

$$\boldsymbol{D}_M^{[\alpha]}(\varphi, \psi) = \boldsymbol{D}_M(\varphi, \psi)+\int_M \int_M \varphi(\xi)U_\alpha(\xi, \eta)\psi(\eta)\mu(d\xi)\mu(d\eta).$$

Let $\mathscr{H}_\alpha$ be the orthogonal complement of $\mathscr{D}^1_{L^2}(D)$ in the Hilbert space

$$(\mathscr{E}^1_{L^2}(D), (\ , \ )_{D,1}+\alpha(\ , \ )_D).$$

The space $(\boldsymbol{H}_M, \boldsymbol{D}_M^{[\alpha]})$ is nothing but the trace on M of the space

$$(\mathscr{H}_\alpha, (\ , \ )_{D,1}+\alpha(\ , \ )_D)\ (^4).$$

EXAMPLE 4(5). Assume that D is bounded. Let $\Delta=\bigcup_{p\in \boldsymbol{P}} E_p$ be a measurable partition of the Martin boundary M. Then, Δ defines a Dirichlet subspace $(\mathscr{F}_M^\Delta, \boldsymbol{D}_M)$ of $(\boldsymbol{H}_M, \boldsymbol{D}_M)$ by $\mathscr{F}_M^\Delta=\{\varphi\in \boldsymbol{H}_M$; there exists a set E_φ such that $\mu'(E_\varphi)=0$ and φ is a constant on E_p-E_φ for each $p\in \boldsymbol{P}\}$. Even when D is a unit disk, $(M, \mu', \mathscr{F}_M^\Delta, \boldsymbol{D}_M)$ is no longer regular except for a trivial case that $\mathscr{F}_M^\Delta$ is equal to $\boldsymbol{H}_M$.

EXAMPLE 5. Consider the whole plane R^2 and put

$$\mathscr{E}(u, v) = \frac{1}{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right)dx\,dy+\frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial u(x, 0)}{\partial x}\frac{\partial v(x, 0)}{\partial x}\,dx,$$

$$\mathscr{A} = \{u\in C_0(R^2);\ u(x, y) \text{ is absolutely continuous in each variable } x \text{ and } y \text{ and } \mathscr{E}(u, u) < +\infty\}.$$

For this space $(\mathscr{A}, \mathscr{E})$, let us check the conditions of Theorem of Appendix. $(\mathscr{A}.1)$ and $(\mathscr{A}.2)$ are evident. To see $(\mathscr{A}.3)$, assume that a sequence $u_n\in\mathscr{A}$ satisfies $(u_n, u_n)_{R^2}\to 0$ and $\mathscr{E}(u_n-u_m, u_n-u_m)\to 0$. We have to prove $\mathscr{E}(u_n, u_n)\to 0$. Since u_n converges to zero in $\mathscr{D}^1_{L^2}(R^2)$ with metric $(\ , \ )_{R^2,1}+(\ , \ )_{R^2}$, we can select a subsequence u_{n_k} such that $u_{n_k}(x, y)$ converges to zero for every (x, y) except on a 2-dimensional Brownian polar set(6) [3]. Especially, $u_{n_k}(x, 0)$ converges to zero for every x except on a set of linear Lebesgue measure zero.

Now it is easy to see that $\int_{-\infty}^{\infty}(\partial u_n(x, 0)/\partial x)^2\,dx\to 0, n\to\infty$. Hence $\mathscr{E}(u_n, u_n)\to 0$ as was to be proved.

(4) Theorem 3.4 of [6].

(5) See footnote 27 of [6].

(6) The subsequent paper [9] will provide general discussions of this point.

By means of Theorem of Appendix we get a D-space $(\mathscr{F}, \mathscr{E})$ on R^2, $(\mathscr{F}, \mathscr{E}^\alpha)$ being the completion of $(\mathscr{A}, \mathscr{E}^\alpha)$ for each $\alpha>0$. This D-space is regular because $C_0^\infty(R^2)\subset\mathscr{A}$ (Corollary 1 of Appendix).

4. **Equivalence of D-spaces.** Consider a D-space $(X, m, \mathscr{F}, \mathscr{E})$. For $u\in L^\infty(X)$ $(=L^\infty(X; m))$, put $\|u\|_\infty=m\text{-ess sup}_{x\in X}|u(x)|$. Let L be a closed subalgebra of $(L^\infty(X), \|\ \|_\infty)$. It is well known that L is then a function lattice and that $u\in L$ implies $u\wedge 1\in L$. Therefore, by making use of Lemma 2.1(iv) and (v), we get the next lemma.

Lemma 4.1. *$\mathscr{F}\cap L$ is a function algebra and a function lattice. Further, $u\in\mathscr{F}\cap L$ implies $u\wedge 1\in\mathscr{F}\cap L$.*

Now we are in a position to define an equivalence relation in the set of all D-spaces.

Definition 4.1. Two D-spaces $(X, m, \mathscr{F}, \mathscr{E})$ and $(X', m', \mathscr{F}', \mathscr{E}')$ are called *equivalent* if there is an algebraic isomorphism Φ from $\mathscr{F}\cap L^\infty(X)$ onto $\mathscr{F}'\cap L^\infty(X')$ and Φ preserves three kinds of metrics: $\|u\|_\infty=\|\Phi u\|'_\infty$, $\mathscr{E}(u, u)=\mathscr{E}'(\Phi u, \Phi u)$ and $(u, u)_X=(\Phi u, \Phi u)_{X'}$ for $u\in\mathscr{F}\cap L^\infty(X)$.

This definition of equivalence is the same as that of [8] where the definition is given in terms of the associated D-rings.

It is not difficult to see that the mapping Φ of Definition 4.1 turns out to be a lattice isomorphism and further Φ can be extended to a unitary map Φ_1 from $(\mathscr{F}, \mathscr{E})$ onto $(\mathscr{F}', \mathscr{E}')$ and a unitary map Φ_2 from $L_0^2(X)$ onto $L_0^2(X')$. Here, $L_0^2(X)$ (resp. $L_0^2(X')$) is the closure of $\mathscr{F}$ (resp. $\mathscr{F}'$) in the metric space $L^2(X)$ (resp. $L^2(X')$). We can use Lemma 2.1(vi) to define the extension Φ_1. The L^2-resolvents $\{G_\alpha, \alpha>0\}$ associated with equivalent D-spaces are mutually related by $G'_\alpha u'=\Phi_2 G_\alpha \Phi_2^{-1}u'$, $u'\in L_0^2(X')$, $\alpha>0$. This relation is proved in [8].

Before proceeding to the next sections, we will summarize here some facts related to Gelfand representations of subalgebras of L^∞. Let (X, m) be as above and L be a closed subalgebra of the real Banach algebra $(L^\infty(X; m), \|\ \|_\infty)$. A nonzero algebraic homomorphism χ from L into real numbers is called a (real) *character* on L. Denote by $\mathscr{M}$ the set of all characters on L. An algebraic homomorph Φ from L into real functions on $\mathscr{M}$ can be defined by

$$\Phi u(\chi) = \chi(u), \qquad u\in L, \quad \chi\in\mathscr{M}. \tag{4.1}$$

We define a neighborhood of $\chi\in\mathscr{M}$ by

$$N(\chi; u_1, u_2, \ldots, u_n; \varepsilon) = \{\chi'\in\mathscr{M}; |\Phi u_k(\chi')-\Phi u_k(\chi)| < \varepsilon, k = 1, 2, \ldots, n\} \tag{4.2}$$

with any $\varepsilon>0$ and $u_1, u_2, \ldots, u_n\in L$. The set $\mathscr{M}$ endowed with topology (4.2) will be called the *character space* of L.

Lemma 4.2. (i) *The character space $\mathscr{M}$ of L is a locally compact Hausdorff space. If the algebra L is countably generated, then $\mathscr{M}$ is separable. $\mathscr{M}$ is compact if and only if $1\in L$.*

(ii) *The map* Φ *of* (4.1) *is an algebraic isomorphism and isometry from* $(L, \| \ \|_\infty)$ *onto* $C(\mathscr{M})$, $C(\mathscr{M})$ *being associated with the uniform norm.*

(iii) *Suppose that* m *is everywhere dense* $L \subset C_b(X)$ (*the space of continuous bounded funtions on* X) *and, for any* $x \in X$, *there is a* $u \in L$ *with* $u(x) \neq 0$. *There exists then a continuous mapping* q *from* X *onto a dense subset of* $\mathscr{M}$ *characterized by*

$$\Phi u(qx) = u(x), \qquad x \in X, \quad u \in L. \tag{4.3}$$

Proof. Consider the space $A = L + (-1)^{1/2}L$ with uniform norm $\| \ \|_\infty$. This is a complex Banach algebra closed under the operation of taking complex conjugate function. If $u \in A$, then

$$\frac{|u|^2}{1+|u|^2} = \frac{1}{1+a^2}\sum_{k=0}^{\infty} |u|^2\left(\frac{a^2-|u|^2}{1+a^2}\right)^k \in L,$$

where $a = \|u\|_\infty$. Therefore, A is a symmetric algebra and the character space $\mathscr{M}$ of L can be identified with the space of regular maximal ideals of A (Loomis [13, subsections 23A and 26C]). Now statements (i) and (ii) of our lemma are known facts. The statement (iii) is evident but we give its proof here for later conveniences. Fix an $x \in X$. A map $u \to u(x)$ is clearly a character on L which we denote by qx. q is continuous at $x \in X$ because any neighborhood $N(\chi; u_1, u_2, \ldots, u_n; \varepsilon)$ of $\chi = qx$ includes the set $q(U(x))$, where $U(x)$ is an open neighborhood of x defined by $U(x) = \{x' \in X; |u_k(x') - u_k(x)| < \varepsilon, k = 1, 2, \ldots, n\}$. Suppose that $q(X)$ is not dense in $\mathscr{M}$. There is then a nonvanishing $v \in C(\mathscr{M})$ such that $v = 0$ on $q(X)$. By (ii) and (4.3), we have $\|v\|_\infty = \|\Phi^{-1}v\|_\infty = \sup_{x \in X} |\Phi^{-1}v(x)| = \sup_{x \in X} |v(qx)| = 0$, which is a contradiction.

Finally we will state the following lemma according to 26J of [13].

LEMMA 4.3. *Suppose that* $\tilde{L}$ *is a dense ideal of* L *and every function in* $\tilde{L}$ *can be expressed as a difference of nonnegative functions in* $\tilde{L}$. *Then, for any positive linear functional* l *on* $\tilde{L}$, *there exists a unique Radon measure* μ *on* $\mathscr{M}$ *such that*

$$\begin{gathered} \Phi(\tilde{L}) \subset L^1(\mathscr{M}; \mu), \\ l(u \cdot v) = \int_{\mathscr{M}} \Phi u(\chi)\Phi v(\chi)\mu(d\chi), \qquad u \in \tilde{L}, \quad v \in L. \end{gathered} \tag{4.4}$$

5. **Regular representations.** Suppose that we are given a D-space $(X, m, \mathscr{F}, \mathscr{E})$. A closed subalgebra L of $L^\infty(X; m)$ will be said *to satisfy condition* (C) if it enjoys the following three properties.

(C.1) L is a countably generated closed subalgebra of $L^\infty(X; m)$.

(C.2) $\mathscr{F} \cap L$ is dense both in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$ and in $(L, \| \ \|_\infty)$, α_0 being a fixed positive number.

(C.3) $L^1(X; m) \cap L$ is dense in $(L, \| \ \|_\infty)$.

THEOREM 2. (i) *There exists at least one* L *satisfying the condition* (C). (ii) *Let an* L *satisfying condition* (C) *be fixed and* X' *be its character space.* X' *is compact if* *and only if* $1 \in L$. *There exists a regular D-space whose underlying space is* X' *and which is equivalent to the given D-space.*

The regular D-space of Theorem 2(ii) will be called a *regular representation of the given D-space with respect to the algebra L.*

Proof of Theorem 2(i). We can find a countable subset D_0 of $C_0(X)$ such that each function in $C_0(X)$ can be uniformly approximated by a sequence of functions in D_0 whose supports are included in a suitable common compactum. D_0 is dense in $L^2(X; m)$. Let $\{G_\alpha, \alpha>0\}$ be the L^2-resolvent associated with the given $(\mathscr{F}, \mathscr{E})$. Then, $G_{\alpha_0}(D_0)\subset\mathscr{F} \cap L^\infty(X; m)$ and $G_{\alpha_0}(D_0)$ is dense in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$ by Lemma 2.1(i). We define L as the closed subalgebra of $L^\infty(X; m)$ generated by $G_{\alpha_0}(D_0)$. It is clear that this L satisfies conditions (C.1) and (C.2). As for (C.3), observe that

$$G_{\alpha_0}(D_0) \subset L^1(X; m) \cap L$$

since

$$\begin{aligned}\int_X |G_{\alpha_0}u|\, dm &\leqq \int_X G_{\alpha_0}|u|\, dm = \sup_{0\leqq v\leqq 1,\, v\in C_0(X)} (v, G_{\alpha_0}|u|)_X \\ &\leqq \frac{1}{\alpha_0}\int_X |u|\, dm < +\infty, \qquad u \in D_0.\end{aligned}$$

Proof of Theorem 2(ii). Let L be a space satisfying condition (C) and X' be its character space. By (C.1) and Lemma 4.2(i), X' is a locally compact Hausdorff and separable space. X' is compact if and only if $1 \in L$. The map Φ of (4.1) is giving an algebraic isomorphism and isometry from L onto $C(X')$. Φ is consequently a lattice isomorph and it holds that $\Phi(u \wedge 1)=(\Phi u)\wedge 1$ for $u \in L$. Let us put

$$\mathscr{R} = \mathscr{F} \cap L, \qquad \mathscr{R}' = \Phi(\mathscr{R}). \tag{5.1}$$

Since $\mathscr{R}$ is dense in L by (C.2), $\mathscr{R}'$ is dense in $C(X')$. Further, by Lemma 4.1, $\mathscr{R}'$ is a lattice and $u' \wedge 1 \in \mathscr{R}'$ whenever $u' \in \mathscr{R}'$.

Keeping these in mind, we are now to construct, step by step, a regular representation $(X', m', \mathscr{F}', \mathscr{E}')$ by making use of the map Φ of (4.1).

(I) *A measure m' on X'.* There exists a unique Radon measure m' on X' which satisfies

$$\begin{aligned}&\Phi(L^1(X; m) \cap L) \subset L^1(X'; m'),\\ &\int_X u(x)v(x)m(dx) = \int_{X'} \Phi u(x')\Phi v(x')m'(dx'), \qquad u \in L^1(X; m) \cap L, \quad v \in L.\end{aligned} \tag{5.2}$$

In fact, by virtue of (C.3), we can apply Lemma 4.3 to a dense ideal $\tilde{L}=L^1(X;m)\cap L$ and a positive linear functional

$$l(u) = \int_X u(x)m(dx), \qquad u \in \tilde{L}.$$

Consider the spaces $\mathscr{R}$ and $\mathscr{R}'$ of (5.1). Since condition (C.2) implies that $\mathscr{R}$ is dense in $\mathscr{F}$ in L^2-sense, we have

$$(5.3) \qquad \mathscr{R} \subset L^2(X; m), \qquad \bar{\mathscr{R}} = L^2_0(X; m),$$

where the closure is taken in L^2-sense and $L^2_0(X; m)$ denotes $\bar{\mathscr{F}}$. Next we will prove

$$(5.4) \qquad \mathscr{R}' \subset L^2(X'; m'), \qquad \bar{\mathscr{R}}' = L^2(X'; m').$$

For any $u \in \mathscr{R}$, $(\Phi u)^2 = \Phi(u^2) \in \Phi(L \cap L^1(X; m))$ and hence $\Phi u \in L^2(X'; m')$ according to (5.2). In order to show that $\mathscr{R}'$ is dense in $L^2(X'; m')$, take a function u in $C_0^+(X')$. Since $\mathscr{R}'$ is uniformly dense in $C(X')$ and is a lattice, we can find a $v \in \mathscr{R}'$ and $u_n \in \mathscr{R}'$ such that $0 \leqq u_n \leqq v$ and u_n converges to u uniformly on X'. Hence, u_n converges to u in $L^2(X'; m')$.

Finally let us show

$$(5.5) \qquad \int_X u(x)v(x)m(dx) = \int_X u'(x')v'(x')m'(dx'), \qquad u, v \in \mathscr{R},$$

where $u' = \Phi u$ and $v' = \Phi v$. Take a nonnegative $v \in \mathscr{R}$. By condition (C.3) and the obvious fact that $L^1(X; m) \cap L$ is a lattice, we can select $v_n \in L^1(X; m) \cap L$ such as $0 \leqq v_n \leqq v$ m-a.e. and $\|v_n - v\|_\infty \to 0$. Since Φ is a lattice isomorph and preserves the uniform norm, the same relations hold for v_n' and v'. Now (5.2) for $u = v = v_n$ leads us to

$$\int_X v(x)^2 m(dx) = \int_{X'} v'(x')^2 m'(dx')$$

which implies (5.5) because each element of $\mathscr{R}$ is expressed as a difference of nonnegative elements of $\mathscr{R}$ and Φ is an algebraic isomorphism.

(II) *Extended map* Φ *on* $L^2_0(X; m)$. In view of (5.3), (5.4), and (5.5) of the preceding paragraph, the algebraic and lattice isomorphism Φ from $\mathscr{R}$ to $\mathscr{R}'$ can be uniquely extended to

(Φ.1) A unitary map Φ from $L^2_0(X; m)$ onto $L^2(X'; m')$.

Let us study the features of this extended map Φ. It has the following properties.

(Φ.2) $L^2_0(X; m)$ is a lattice and Φ is a lattice isomorphism. $\Phi(u \wedge 1) = (\Phi u) \wedge 1$ whenever $u \in L^2_0(X; m)$.

(Φ.3) Φ is an algebraic isomorphism from $L^2_0(X; m) \cap L^\infty(X; m)$ onto

$$L^2(X'; m') \cap L^\infty(X'; m').$$

Further it holds that

$$(5.6) \qquad \|u\|_\infty = \|\Phi u\|'_\infty, \qquad u \in L^2_0(X; m) \cap L^\infty(X; m).$$

To prove (Φ.2), take a $u \in L^2_0(X; m)$ and find a sequence $u_n \in \mathscr{R}$ which converges to u in L^2-sense. Since $|u_n| \in \mathscr{R}$ converges to $|u|$ in L^2-sense, $|u| \in L^2_0(X; m)$. Since Φ is a lattice isomorph on $\mathscr{R}$ and preserves L^2-norm, we have $\Phi|u| = \text{l.i.m. } \Phi|u_n| = \text{l.i.m. } |\Phi u_n| = |\Phi u|$. Thus we have proved the first half of (Φ.2). The latter half is similarly proved.

The property (Φ.3) follows from (Φ.2). In fact, for $u \in L_0^2(X; m)$ with $\|u\|_\infty = a < +\infty$, we have $|\Phi u| = \Phi(|u|) = \Phi(|u| \wedge a) = |\Phi u| \wedge a$ which means $\|\Phi u\|'_\infty \leqq \|u\|_\infty$. In the same way, we have $\|u'\|_\infty \geqq \|\Phi^{-1}u'\|_\infty$ for $u' \in L^2(X'; m') \cap L^\infty(X'; m')$. To see that Φ is an algebraic isomorphism, take a $u \in L_0^2(X; m) \cap L^\infty(X; m)$ and a sequence $u_n \in \mathscr{R}$ which converges to u in L^2-sense. We may assume that $|u_n| \leqq \|u\|_\infty$. Then u_n^2 (resp. $(\Phi u_n)^2$) converges to u^2 (resp. $(\Phi u)^2$) in L^2-sense. Since Φ is an algebraic isomorph on $\mathscr{R}$, $\Phi(u^2) = \text{l.i.m.}\ \Phi(u_n^2) = (\Phi u)^2$.

(III) *Induced D-space* $(X', m', \mathscr{F}', \mathscr{E}')$. By means of the preceding map Φ on $L_0^2(X; m) \supset \mathscr{F}$, we define

$$\mathscr{F}' = \Phi(\mathscr{F}), \tag{5.7}$$
$$\mathscr{E}'(u', v') = \mathscr{E}(\Phi^{-1}u', \Phi^{-1}v'), \qquad u', v' \in \mathscr{F}'.$$

Then, $(X', m', \mathscr{F}', \mathscr{E}')$ is a D-space.

Condition (D.1) for (X', m') has already been proved and (D.2) for $(\mathscr{F}', \mathscr{E}')$ is obvious by the property (Φ.1) of Φ. Instead of proving (D.3), let us check an equivalent condition (D.3)$'$ in Remark 2.1. Take $u' \in \mathscr{F}'$ and put $v' = (0 \vee u') \wedge 1$, $u = \Phi^{-1}u'$. Then we have $v' = 0 \vee \Phi u \wedge 1 = \Phi(0 \vee u \wedge 1)$ by (Φ.2). Since $v = 0 \vee u \wedge 1 \in \mathscr{F}$ and $\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$, $v' \in \mathscr{F}'$ and $\mathscr{E}'(v', v') \leqq \mathscr{E}'(u', u')$ proving (D.3)$'$.

(IV) $(X', m', \mathscr{F}', \mathscr{E}')$ *is equivalent to* $(X, m, \mathscr{F}, \mathscr{E})$. This is evident from ($\Phi$.1), ($\Phi$.3) and (5.7).

(V) $(X', m', \mathscr{F}', \mathscr{E}')$ *is regular.* Φ preserves $\mathscr{E}^{\alpha_0}$-norm and the uniform norm on $\mathscr{R} = \mathscr{F} \cap L$. Hence by virtue of condition (C.2), $\mathscr{R}' = \Phi(\mathscr{R})$ is dense both in $\mathscr{F}'$ and in $C(X')$. Since $\mathscr{R}$ is the intersection of $\mathscr{F}$ and the uniform closure of $\mathscr{R}$, the same relation holds for $\mathscr{R}'$ and $\mathscr{F}'$. Therefore

$$\mathscr{R}' = \mathscr{F}' \cap C(X'). \tag{5.8}$$

On the other hand we have by (5.6),

$$\sup_{x' \in X'} |u'(x')| = m'\text{-ess}\sup_{x' \in X'} |u'(x')|, \qquad u' \in \mathscr{F}' \cap C(X'). \tag{5.9}$$

Since $\mathscr{F}' \cap C(X')$ is dense in $C(X')$, (5.9) means that m' is everywhere dense on X'. The proof of (V) is complete.

The proof of Theorem 2 has ended.

The next remarks and lemma will state the meaning of Theorem 2 for special cases.

REMARK 5.1. Suppose that the given D-space $(X, m, \mathscr{F}, \mathscr{E})$ is regular. Since m is everywhere dense, $C(X)$ may be considered as a closed subalgebra of $L^\infty(X; m)$. Obviously $C(X)$ satisfies conditions (C.1) and (C.2). It also satisfies (C.3) because of $L^1(X; m) \cap C(X) \supset C_0(X)$. Therefore, we may consider the regular representation with respect to $C(X)$. However, as is well known, the character space of $C(X)$ coincides with X itself, and after all the regular representation goes back to the given regular D-space without any change.

LEMMA 5.1. *Suppose that m is everywhere dense. Suppose further that an algebra L satisfies not only conditions* (C.1), (C.2) *and* (C.3) *but also the following.*

(C.4) *$L \subset C_b(X)$, L separates points of X and, at any $x \in X$, there is a $u \in L$ such that $u(x) \neq 0$.*

Let $(X', m', \mathscr{F}', \mathscr{E}')$ be the regular representation with respect to this L. Then,

(i) *X is continuously embedded onto a dense subset of X'. By this embedding, any Borel set of X goes to a Borel set of X' and the restriction to X of any Borel set of X' is a Borel set of X (with respect to the original topology).*

(ii) *For any Borel subset A of X', $m'(A) = m(A \cap X)$. Therefore, the space $(L^2(X'; m'), (\ ,\ )_{X'})$ is identified with the space $(L^2(X; m), (\ ,\ )_X)$.*

(iii) *By the above identification, $(\mathscr{F}', \mathscr{E}')$ is equal to $(\mathscr{F}, \mathscr{E})$.*

Proof. By virtue of (C.4), the map q of (4.3) from X onto a dense subset of X' is not only continuous but also one-to-one. The rest of the lemma is obvious.

REMARK 5.2. Consider the situation of Example 1 of §3. If ∂D is of class C^1, then the space $L = \{u \in C_b(D);\ u \text{ is continuously extendable to } \bar{D}\}$ satisfies conditions (C.1)$\sim$(C.4). $\{\bar{D}, dx, \mathscr{E}^1_{L^2}, (\ ,\ )_{D,1}\}$ is just the regular representation of $\{D, dx, \mathscr{E}^1_{L^2}, (\ ,\ )_{D,1}\}$ with respect to this L. In this case, D is homeomorphically embedded into $\bar{D}$. Coming back to the general case of Lemma 5.1, X is homeomorphically embedded onto a dense subset of X' if and only if for any $x_0 \in X$ and $Y \subset X$ such as $x_0 \notin \bar{Y}$, there exists $u \in L$ such that $u(x_0) = 1$ and $u(x) = 0$ on Y.

6. **Strongly regular representations.** Suppose that we are given a D-space $(X, m, \mathscr{F}, \mathscr{E})$. Denote by $\{G_\alpha, \alpha > 0\}$ its associated L^2-resolvent.

LEMMA 6.1. (i) *G_α makes the space $L^2(X; m) \cap L^\infty(X; m)$ invariant and*

$$\|G_\alpha u\|_\infty \leqq \frac{1}{\alpha} \|u\|_\infty, \qquad u \in L^2 \cap L^\infty. \tag{6.1}$$

(ii) *G_α makes the space $L^\infty(X; m) \cap L^1(X; m)$ $(\subset L^2(X; m))$ invariant and*

$$\int_X |G_\alpha u(x)| m(dx) \leqq \frac{1}{\alpha} \int_X |u(x)| m(dx), \qquad u \in L^\infty \cap L^1. \tag{6.2}$$

Inequality (6.2) for $u \in C_0(X)$ has already been proved in the proof of Theorem 2(i). The proof for $u \in L^\infty \cap L^1$ is the same. The rest of Lemma 6.1 is clear.

Owing to Lemma 6.1(i), G_α on $L^2 \cap L^\infty$ can be uniquely extended to a linear operator $\bar{G}_\alpha$ on $L^\infty_0(X; m)$ (the closure of $L^2 \cap L^\infty$ in L^∞). $\{\bar{G}_\alpha, \alpha > 0\}$ is a *sub-Markov resolvent on L^∞_0*, that is,

($\bar{\text{G}}$.1) If $u \in L^\infty_0$ and $0 \leqq u \leqq 1$ m-a.e. then $0 \leqq \alpha \bar{G}_\alpha u \leqq 1$ m-a.e.

$$\bar{G}_\alpha - \bar{G}_\beta + (\alpha - \beta)\bar{G}_\alpha \bar{G}_\beta = 0, \qquad \alpha, \beta > 0. \tag{$\bar{\text{G}}$.2}$$

A closed subalgebra L of $L^\infty_0(X; m)$ is said *to satisfy condition* (R) if it enjoys the following two properties.

(R.1) $\bar{G}_\alpha(L) \subset L$ for every $\alpha > 0$.

(R.2) L is generated by a countable subset L_0 of $\mathscr{F} \cap L$ such that each $u \in L_0$ is nonnegative and satisfies $\alpha \bar{G}_{\alpha+\alpha_0} u \leqq u$, m-a.e., $\alpha > 0$.

THEOREM 3. (i) *There exists an* L *satisfying condition* (R) *as well as* (C). (ii) *Fix an* L *which satisfies* (C) *and* (R). *The regular representation of the given D-space with respect to this* L *turns out to be strongly regular.*

We need the next lemma for the proof of Theorem 3(i).

LEMMA 6.2. *Let* S_0 *be a set of countable nonnegative functions in* $\mathscr{F} \cap L^\infty \cap L^1$. *Then, there exists a set* S *possessing the following features.*

(S.1) $S \supset S_0$ *and* S *is a countably generated subalgebra of* $\mathscr{F} \cap L^\infty \cap L^1$. *Each function of* S *is expressed as a difference of nonnegative functions of* S.

(S.2) *For any* $\alpha > 0$, $\bar{G}_\alpha$ *makes the space* $\bar{S}$ *invariant,* $\bar{S}$ *being the closure of* S *in* L^∞.

Proof. According to F. Knight [11, Lemma 1], we construct S as follows. Starting with S_0, assume $S_1, \ldots, S_n$ are defined. Define S_{n+1} as an algebra generated by $\{S_n, G_{\alpha_1}(S_n), \ldots, G_{\alpha_n}(S_n), G_{\alpha_{n+1}}(S_n)\}$, where $\{a_k\}$ is the set of all positive rational numbers. Put $S = \bigcup_{n=0}^\infty S_n$, which satisfies condition (S.1) by virtue of Lemma 6.1 and of the fact that $\mathscr{F} \cap L^\infty \cap L^1$ is an algebra (Lemma 4.1). It is easy to see that condition (S.2) is met.

Proof of Theorem 3(i). Let D_0^+ be a countable subset of $C_0^+(X)$ such that the set $D_0 = \{u = u_1 - u_2;\ u_i \in D_0^+,\ i = 1, 2\}$ has the property in the proof of Theorem 2(i). Put $S_0 = G_{\alpha_0}(D_0^+)$, which satisfies the following.

(S_0.1) S_0 is a countable set of nonnegative functions in $\mathscr{F} \cap L^\infty \cap L^1$.

(S_0.2) The set $\{u = u_1 - u_2;\ u_i \in S_0,\ i = 1, 2\}$ is dense in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.

(S_0.3) $\alpha G_{\alpha+\alpha_0} u \leqq u$ m-a.e. for $u \in S_0$ and $\alpha > 0$.

For such an S_0, let S be a set which satisfies conditions (S.1) and (S.2) of Lemma 6.2. By (S.1), there exists a set $\tilde{S}$ of countable nonnegative functions in S whose linearization is just S. Let us put

$$L_0 = S_0 \cup G_{\alpha_0}(\tilde{S}), \tag{6.3}$$

$$L = \text{the closed subalgebra of } L^\infty \text{ generated by } L_0, \tag{6.4}$$

then the space L meets both conditions (C) and (R).

In order to check condition (C) of §5, denote by $\mathscr{R}_0$ the algebra generated by L_0. By (S_0.1), (S.1) and Lemma 6.1, L_0 and hence $\mathscr{R}_0$ are included in $\mathscr{F} \cap L^\infty \cap L^1$. Notice that $\mathscr{R}_0 \subset \mathscr{F} \cap L$ and that L is the closure of $\mathscr{R}_0$ in L^∞. Therefore both $\mathscr{F} \cap L$ and $L^1(X; m) \cap L$ are dense in L. Since $\mathscr{R}_0$ contains the set of (S_0.2), $\mathscr{F} \cap L$ is dense in $(\mathscr{F}, \mathscr{E}^{\alpha_0})$.

Coming to condition (R), it is clear that condition (R.2) is satisfied by L_0 of (6.3). Observe that L is the closed subalgebra of L^∞ generated by $S_0 \cup \bar{G}_{\alpha_0}(\bar{S})$

By conditions (S.1) and (S.2), this means $L \subset \bar{S}$ and hence $\bar{G}_\alpha(L) \subset \bar{G}_\alpha(\bar{S}) = \bar{G}_{\alpha_0}(\bar{S}) \subset L$ proving property (R.1) for L.

Proof of Theorem 3(ii). Let us fix an L which satisfies conditions (C) and (R) and let $(X', m', \mathscr{F}', \mathscr{E}')$ be the regular representation with respect to L according to Theorem 2(ii). We have to prove that $(\mathscr{F}', \mathscr{E}')$ is generated by a Ray resolvent kernel on X' and $\mathscr{F}' \cap C(X')$ contains a set C_1' attached to the Ray resolvent (Definition 2.5).

A Ray resolvent can be constructed by Φ of (4.1) which is an algebraic isomorph and isometry from L onto $C(X')$. Φ is a lattice isomorph and satisfies $\Phi(u \wedge 1) = (\Phi u) \wedge 1$ for $u \in L$. Indeed,

$$\bar{G}_\alpha' u' = \Phi \bar{G}_\alpha \Phi^{-1} u', \qquad u' \in C(X'), \quad \alpha > 0, \tag{6.5}$$

$$C_1' = \Phi(L_0) \tag{6.6}$$

define a Ray resolvent operator $\{\bar{G}_\alpha', \alpha > 0\}$ on $C(X')$ and a set C_1' attached to it.

$\bar{G}_\alpha'$ is a sub-Markov resolvent on $C(X')$ on account of (R.1) for L and ($\bar{\mathrm{G}}$.1), ($\bar{\mathrm{G}}$.2) for $\bar{G}_\alpha$ on L_0^∞. (R.2) implies that C_1' generates the closed algebra $C(X')$ and so that C_1' separates points of X' and, for any $x' \in X'$, there exists $u' \in C_1'$ nonvanishing at x'. The inequalities $u' \geqq 0$, $\alpha \bar{G}_{\alpha+\alpha_0} u' \leqq u'$ for $u' \in C_1'$ are obvious from (R.2).

We see that C_1' is included in $\mathscr{F}' \cap C(X')$ because of (5.8) and (R.2).

Finally, let us prove that $\{\bar{G}_\alpha', \alpha > 0\}$ generates the space $(\mathscr{F}', \mathscr{E}')$. It suffices to show

$$\bar{G}_\alpha' u = G_\alpha' u, \quad m'\text{-a.e.}, \qquad u \in L^2(X'; m') \cap C(X'), \tag{6.7}$$

where $\{G_\alpha', \alpha > 0\}$ is the L^2-resolvent associated with $(\mathscr{F}', \mathscr{E}')$.

Observe that G_α' is related to the L^2-resolvent G_α associated with $(\mathscr{F}, \mathscr{E})$ as follows.

$$G_\alpha' u' = \Phi_2 G_\alpha \Phi_2^{-1} u', \qquad u' \in L^2(X'; m'). \tag{6.8}$$

Here, Φ_2 denotes the unitary map from $L_0^2(X; m)$ onto $L^2(X'; m')$ as appeared in step (II) of the proof of Theorem 2(ii). We have indeed by (5.7), $\mathscr{E}'^\alpha(G_\alpha' u', v') = (u', v')_{X'} = (\Phi_2^{-1} u', \Phi_2^{-1} v')_X = \mathscr{E}^\alpha(G_\alpha \Phi_2^{-1} u', \Phi_2^{-1} v') = \mathscr{E}'^\alpha(\Phi_2 G_\alpha \Phi_2^{-1} u', v')$ for any $v' \in \mathscr{F}'$.

Since Φ and Φ_2 coincide on $\mathscr{F} \cap L$ and $\bar{G}_\alpha$ is equal to G_α on $\mathscr{F} \cap L$, (6.5) and (6.8) lead us to the equality (6.7) for $u' \in \mathscr{F}' \cap C(X')$. However $\mathscr{F}' \cap C(X')$ is dense in $C(X')$. Therefore, taking sub-Markovity of $\bar{G}_\alpha'$ and G_α' into account, we get (6.7) for $u' \in L^2(X'; m') \cap C(X')$.

The proof of Theorem 3 is complete.

The next lemma expresses the meaning of Theorem 3 for a special case.

LEMMA 6.1. *Suppose that m is everywhere dense. Suppose further that the next condition is satisfied.*

(G.3) *$(\mathscr{F}, \mathscr{E})$ is generated by a symmetric resolvent kernel $\{\tilde{G}_\alpha, \alpha>0\}$ on X such that $\tilde{G}_\alpha$ transforms $C_b(X)$ into $C_b(X)$ and* $\lim_{\alpha\to+\infty} \alpha G_\alpha u(x)=u(x)$ *for any $x\in X$, $u\in C_b(X)$.*

(i) *There exists then an algebra L which satisfies not only* (C) *and* (R) *but also the additional condition* (C.4) *of Lemma* 5.1.

(ii) *Let $(X', m', \mathscr{F}', \mathscr{E}')$ be the regular representation with respect to such an L. Then, this is strongly regular and X is embedded onto a dense subset of X' in such a way as Lemma* 5.1. *The associated Ray resolvent kernel $\bar{G}'_\alpha$ on X' is an extension of $\tilde{G}_\alpha$ of* (G.3) *in the following sense. For any Borel set A of X,*

$$\bar{G}'_\alpha(x, A) = \tilde{G}_\alpha(x, A), \qquad x\in X. \tag{6.9}$$

Proof. (i) By replacing L^2-resolvent $\{G_\alpha\}$ with the smooth resolvent $\{\tilde{G}_\alpha\}$ of (G.3), we can repeat the arguments of the proof of Theorem 3(i) to get an L in $C_b(X)$. Moreover, S_0 $(\subset L)$ separates points of X. In fact, assume that $\tilde{G}_{\alpha_0}u(x) =\tilde{G}_{\alpha_0}u(y)$ for every $u\in D_0^+$. Then, it is valid for $u\in C_b(X)$. Hence $\alpha\tilde{G}_\alpha u(x)=\alpha\tilde{G}_\alpha u(y)$ for all $\alpha>0$ and $u\in C_b(X)$. By letting α tend to infinity, we have $u(x)=u(y)$, $u\in C_b(X)$, which means $x=y$. In the same way, we see the existence of some function of S_0 nonvanishing at any preassigned point of X.

(ii) The identity (6.8) is equivalent to

$$\bar{G}'_\alpha u'(x) = \tilde{G}_\alpha u(x), \qquad u'\in C(X'), \quad x\in X, \tag{6.10}$$

where $u=u'|_X$ the restriction of u' to X. The right-hand side of (6.10) makes sense because $u\in C_b(X)$. Since (4.3) implies $u'|_X=\Phi^{-1}u'$ for any $u'\in C(X')$, we have

$$\bar{G}'_\alpha u'|_X = \Phi^{-1}\bar{G}'_\alpha u' = \Phi^{-1}\Phi\bar{G}_\alpha\Phi^{-1}u' = \bar{G}_\alpha u, \qquad u'\in C(X'), \quad \text{by (6.5)}.$$

However, $\bar{G}_\alpha$ and $\tilde{G}_\alpha$ are identical on L for they are on $L^2(X; m)\cap C(X)$.

REMARK 6.1. We may consider that Theorem 3 treats the problem of finding strong Markov processes for a given resolvent operator. Theorem 3 solves this problem demanding that the construction procedure does not change the structure of certain associated function spaces. If we take off such a demand, we have much more possibilities of getting strong Markov processes. The proof of Theorem 3 indicates the following.

Suppose that we are given a sub-Markov resolvent operator $\{\bar{G}_\alpha, \alpha>0\}$ on a closed subalgebra A of $B(X)$ or $L^\infty(X; m)$. Here, $B(X)$ denotes the space of bounded functions with uniform norm. No kind of assumption of symmetry is imposed on $\bar{G}_\alpha$.

(I) If we are given a closed subalgebra L of A which satisfies condition (R)[7], then (6.5) defines a Ray resolvent (and consequently a strong Markov process of Ray in the sense of Remark 2.2(ii)) on the very character space X' of L.

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[7] Here the term of condition (R) is used under a trivial modification that we do not require L_0 of (R.2) to be a subset of $\mathscr{F}$.

(II) Let D_0^+ be any countable subcollection of A^+. Then D_0^+ generates an L satisfying condition (R) quite in the same manner as in the proof of Theorem 3.

Our method to get L which satisfies (R) is due to H. Kunita and T. Watanabe [12]. The above mentioned facts tell the generality of their method and the scope of the Ray process.

REMARK 6.2. Consider a bounded domain D of R^N. The D-space of Example 1 of §3 meets the condition (G.3) of Lemma 6.1. According to Lemma 6.1, we get its strongly regular representation accompanied by a Ray process on an extension D' of D. On the other hand, we adopted in [5] the compactification D^* of D with respect to $G_1(D_0^+)$ to serve as a state space of an extended strong Markov process—a reflecting Brownian motion. This process is not necessarily a Ray's one in the strict sense of the word. However, it turns out that $(D^*, dx, \mathscr{E}_{L^2}^1, (\ ,\ )_{D,1})$ is a regular representation of the given D-space, for the algebra generated by $G_1(D_0^+)$ and 1 is obviously dense both in $C(D^*)$ and in $\mathscr{E}_{L^2}^1$.

The situation is quite the same for the D-space generated by each resolvent density of class $\boldsymbol{G}$ in [6].

Appendix. *Construction of D-spaces by means of completion.* Let X be a locally compact Hausdorff and separable space and m be a Radon measure on X. A pair $(\mathscr{A}, \mathscr{E})$ is said to satisfy condition $(\mathscr{A})$ if it enjoys the next three conditions.

$(\mathscr{A}.1)$ $\mathscr{A}$ is a linear subspace of $L^2(X; m)$ and $\mathscr{E}$ is a positive definite symmetric bilinear form on $\mathscr{A}$.

$(\mathscr{A}.2)$ If $u \in \mathscr{A}$, then $v=(0 \vee u) \wedge 1 \in \mathscr{A}$ and $\mathscr{E}(v, v) \leqq \mathscr{E}(u, u)$.

$(\mathscr{A}.3)$ If $u_n \in \mathscr{A}$ satisfies $(u_n, u_n)_X \to 0$ and $\mathscr{E}(u_n-u_m, u_n-u_m) \to 0$, then

$$\mathscr{E}(u_n, u_n) \to 0.$$

Condition $(\mathscr{A}.1)$ means that $\mathscr{A}$ is a real pre-Hilbert space with respect to inner product $\mathscr{E}^\alpha(u, v)=\mathscr{E}(u, v)+\alpha(u, v)_X$, $u, v \in \mathscr{A}$, for each $\alpha>0$.

THEOREM. *Suppose that a pair $(\mathscr{A}, \mathscr{E})$ satisfies condition $(\mathscr{A})$. Let $\mathscr{F}$ be the completion of $\mathscr{A}$ with respect to a metric $\mathscr{E}^{\alpha_0}$ for a fixed $\alpha_0>0$. Then, $(X, m, \mathscr{F}, \mathscr{E})$ is a D-space.*

Proof. $(\mathscr{A}.1)$ and $(\mathscr{A}.3)$ imply that $\mathscr{F}$ is a linear subspace of $L^2(X; m)$ and that $(\mathscr{F}, \mathscr{E})$ satisfies the condition (D.2) of Definition 2.1. Therefore, for each $\alpha>0$ and $u \in L^2(X; m)$, there exists $G_\alpha u \in \mathscr{F}$ such that $\mathscr{E}^\alpha(G_\alpha u, v)=(u, v)_X$ holds for any $v \in \mathscr{F}$. It suffices for us to show that $\{G_\alpha, \alpha>0\}$ is an L^2-resolvent, because then $(\mathscr{F}, \mathscr{E})$ coincides with the D-space generated by $\{G_\alpha, \alpha>0\}$. Obviously $\{G_\alpha, \alpha>0\}$ satisfies the resolvent equation. To see its sub-Markov property, let us assume that $u \in L^2(X; m)$ and $0 \leqq u \leqq 1$ m-a.e.

If we put $\Phi(v)=\mathscr{E}(v, v)+\alpha(v-(1/\alpha)u, v-(1/\alpha)u)_X$ for $v \in \mathscr{F}$, then we have $\Phi(v)=\Phi(G_\alpha u)+\mathscr{E}^\alpha(G_\alpha u-v, G_\alpha u-v)$, which means that $G_\alpha u$ is a unique element of $\mathscr{F}$ minimizing the quadratic form Φ on $\mathscr{F}$. Further we see that $v_n \in \mathscr{F}$ converges to $G_\alpha u$ in $\mathscr{E}^\alpha$-norm if and only if v_n is a minimizing sequence for Φ: $\Phi(v_n) \to \Phi(G_\alpha u)$.

Since $\mathscr{A}$ is dense in $\mathscr{F}$ in $\mathscr{E}^\alpha$-norm, there exist $v_n \in \mathscr{A}$ which converges to $G_\alpha u$ in $\mathscr{E}^\alpha$-norm. Put $w_n = (0 \vee v_n) \wedge (1/\alpha)$. By condition ($\mathscr{A}$.2), $w_n \in \mathscr{A}$ and $\mathscr{E}(w_n, w_n) \leqq \mathscr{E}(v_n, v_n)$. Now it is easy to see that $\Phi(G_\alpha u) \leqq \Phi(w_n) \leqq \Phi(v_n)$ for each n. However, v_n is a minimizing sequence for Φ and so that w_n is. Hence, w_n converges to $G_\alpha u$ in $\mathscr{E}^\alpha$-norm and consequently a subsequence of w_n converges to $G_\alpha u$ m-a.e. Thus we get $0 \leqq G_\alpha u \leqq 1/\alpha$ m-a.e.

COROLLARY 1. *In addition to the condition in Theorem, we assume that m is everywhere dense on X and that $\mathscr{A}$ is a dense subset of $C(X)$. Then $(X, m, \mathscr{F}, \mathscr{E})$ of the theorem is a regular D-space.*

COROLLARY 2. *Suppose that we are given a D-space $(X, m, \mathscr{F}, \mathscr{E})$. Let $\mathscr{A}$ be a subspace of $\mathscr{F}$ such that $(0 \vee u) \wedge 1 \in \mathscr{A}$ whenever $u \in \mathscr{A}$. Denote by $\mathscr{F}_0$ the completion of $\mathscr{A}$ with respect to $\mathscr{E}^{\alpha_0}$-norm. Then, $(X, m, \mathscr{F}_0, \mathscr{E})$ is a D-space.*

COROLLARY 3. *Suppose that we are given a D-space $(X, m, \mathscr{F}, \mathscr{E})$ with everywhere dense m. We assume that $\mathscr{F} \cap C_0(X)$ is dense in $C_0(X)$. Denote by $\mathscr{F}_0$ the completion of $\mathscr{F} \cap C_0(X)$ with respect to $\mathscr{E}^{\alpha_0}$-norm. Then, $(X, m, \mathscr{F}_0, \mathscr{E})$ is a regular D-space.*

REFERENCES

1. A. Beurling and J. Deny, *Dirichlet spaces*, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 208–215. MR **21** #5098.

2. J. Deny, *Principe complet du maximum et contractions*, Ann. Inst. Fourier (Grenoble) **15** (1965), fasc. 1, 259–272. MR **32** #5913.

3. J. Deny and J. L. Lions, *Les espaces du type de Beppo Levi*, Ann. Inst. Fourier (Grenoble) **5** (1953/54), 305–370. MR **17**, 646.

4. J. L. Doob, *Boundary properties of functions with finite Dirichlet integrals*, Ann. Inst. Fourier (Grenoble) **12** (1962), 573–621. MR **30** #3992.

5. M. Fukushima, *A construction of reflecting barrier Brownian motions for bounded domains*, Osaka J. Math. **4** (1967), 183–215. MR **37** #6999.

6. ———, *On boundary conditions for multi-dimensional Brownian motions with symmetric resolvent densities*, J. Math. Soc. Japan **21** (1969), 58–93. MR **38** #5291.

7. ———, *Dirichlet spaces and their representations*, Seminar on Probability **31** (1969). (Japanese)

8. ———, *On Dirichlet spaces and Dirichlet rings*, Proc. Japan Acad. **45** (1969), 433–436.

9. ———, *Dirichlet spaces and strong Markov processes* (to appear).

10. I. M. Gel'fand, D. A. Raĭkov and G. E. Šilov, *Commutative normed rings*, Fizmatgiz, Moscow, 1960; English transl., Chelsea, New York, 1964. MR **23** #A1242; MR **34** #4940.

11. F. Knight, *Note on regularization of Markov processes*, Illinois J. Math. **9** (1965), 548–552. MR **31** #1713.

12. H. Kunita and T. Watanabe, *Some theorems concerning resolvents over locally compact spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Prob. (Berkeley, Calif., 1965/66), vol. II: Contributions to Probability Theory, part 2, Univ. of California Press, Berkeley, Calif., 1967, pp. 131–164. MR **35** #4999.

13. L. H. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, Princeton, N. J., 1953. MR **14**, 883.

14. S. Mizohata, *The theory of partial differential equations*, Contemporary Math., no. 9, Iwanami Shoten, Tokyo, 1965. (Japanese) MR **38** #396.

15. D. Ray, *Resolvents, transition functions, and strongly Markovian processes*, Ann. of Math. (2) **70** (1959), 43–72. MR **21** #6027.

16. T. Shiga and T. Watanabe, *On Markov chains similar to the reflecting barrier Brownian motion*, Osaka J. Math. **5** (1968), 1–33. MR **39** #7676.

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